A counterexample for Barkatou's conjecture on the exponential growth order of solutions for Moser irreducible system and surgery operations (Algebraic analytic methods in complex partial differential equations)

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A counterexample for Barkatou's conjecture on the exponential growth order of solutions for Moser irreducible system and surgery operations

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Abstract

We, first, give a general result on the estimate of maximal exponential growth order $\rho(L)$ of solutions of $Ly=0$ with $L=z^{p+1}(d/dz)I_N-A(z)$ ($A(z)\in M_N(\mathbf{C}\{z\})$) which is Moser irreducible defined by J. Moser [Mos]. Next, we give a counterexample for the conjecture by M. Barkatou on the characterization of $\rho(L)$ stated in his lecture [Bar]. We, further, introduce a class of system transformations called surgery operations by which the leading term of exponential factor of the formal fundamental matrix solution is calculated exactly for the obtained counterexample.

1 Introduction

A singular system $L=(p,A(z))$ of apparent Poincaré rank $p\geq 1$ is defined by

$$(1.1)\qquad L\equiv(p,A(z)):=z^{p+1}\frac{d}{dz}I_N-A(z),\quad A(z)=\big(a_{ij}(z)\big)\in M_N(\mathbb{C}\{z\}).$$

We denote by $\rho(L)\in\mathbb{Q}_{\geq 0}$ the maximal exponential growth order in $|z|^{-1}$ of solutions $y(z)$ of the homogeneous equation $Ly(z)=0$, which we call the *irregularity* of L. The case $\rho(L)=0$ is understood that the system L is regular singular at $z=0$.

The characterization of $\rho(L)$ were established in [Kit] and [M-I] (also in [Miy2]) by reducing the system $L=(p,A(z))$ into a non-degenerate system in Volevič's sense.

In this paper we shall give a little more direct estimate for $\rho(L)$ for Moser irreducible system case $L=(p,A(z))$, which is defined as follows (cf. [Mos]).

Let $A(z)=\sum_{n=0}^{\infty}A_nz^{k+n}$ ($A_0\neq O$) be the Taylor expansion of $A(z)$, where $k=O(A)\geq 0$ denotes the order of zeros of $A(z)$ at $z=0$. Then he defined two numbers,

$$(1.2)\qquad \begin{aligned}&m(A):=p-k+r/N\quad(r=\operatorname{rank}A_0)\qquad\qquad(\text{Moser's rank}),\\&\mu(A):=\min_{P(z)\in GL_N(\mathbb{K}[z])}\{m(A_P)\ ;\ A_P(z)\in M_N(\mathbb{C}\{z\})\}\quad(\text{reduced Moser's rank}),\end{aligned}$$

where $A_P(z)$ denotes the reduced matrix of $A(z)$ by an invertible matrix $P(z)$ over $\mathbb{K}[z]$ (the field of fractions of $\mathbb{C}[z]$),

$$(1.3)\qquad A_P(z):=P^{-1}(z)A(z)P(z)-z^{p+1}P^{-1}(z)P'(z).$$

Our interest is in the case when $m(A) > 1$, since L is regular singular at $z = 0$ if $m(A) \leq 1$. Then he defined

Definition ((ir)reducibility) Let $m(A) > 1$. Then the system $L = (p, A(z))$ is called Moser reducible if $m(A) > \mu(A)$. Otherwise, it is called Moser irreducible.

In order to characterize the reducible system, he introduced a kind of characteristic polynomial $\mathcal{P}_A(\lambda) \in \mathbb{C}[\lambda]$, which we call *Moser's polynomial*, by

$$(1.4) \quad \mathcal{P}_A(\lambda) := z^r \times \det\left(\lambda I_N - A(z)/z^{k+1}\right)\big|_{z=0} = z^r \times \det\left(\lambda I_N - (A_0/z + A_1)\right)\big|_{z=0}.$$

Then he proved

Theorem 1.1 [Mos] *Let a system $L = (p, A(z))$ satisfy $m(A) > 1$. Then L is Moser reducible if and only if $\mathcal{P}_A(\lambda) \equiv 0$.*

Under these preparations, a characterization theorem of $\rho(L)$ is obtained in the form,

Theorem 1.2 *Let $L = (p, A(z))$ be Moser irreducible with non-zero nilpotent constant term $A_0 = A(0)$ $(k = 0)$, and define that $r = \mathrm{rank} A_0$, $k_1 = \min\{k \geq 2\ ;\ A_0^k = O\}$ and $d = \deg_\lambda \mathcal{P}_A \geq 0$. Then $\rho(L)$ is estimated by*

$$(1.5) \qquad p - S_0(L) \leq \rho(L) \leq p - 1/k_1, \quad S_0(L) := (N - d - r)/(N - d).$$

Remark (i) The first inequality in (1.5) is found in [Bar, p.31] without proof.
(ii) The both inequalities in (1.5) are best possible. Indeed, in each inequality the equality is actually attained by an example which we omit.

Here is the conjecture by Barkatou.

Conjecture [Bar, p.35] *Let $L = (p, A(z))$ be a Moser irreducible system as in Theorem 1.2. Let $p_A(z, \lambda) := \det(\lambda I_N - A(z)) = \sum_{j=0}^N p_j(z)\lambda^{N-j}$ be the characteristic polynomial of $A(z)$. Then the following equality may hold*

$$(1.6) \qquad \rho(L) = p - s_0(A), \quad s_0(A) := \min\{O(p_j)/j\ ;\ 1 \leq j \leq N\} > 0,$$

where $O(p_j)$ denotes the order of zeros of $p_j(z)$ at $z = 0$ and $s_0(A)$ gives the minimal slope of sides of the Newton polygon of $A(z)$ defined later.

The conjecture does hold trivially when $s_0(A) = 0$, but it is not correct in general which is shown by a counterexample in Section 3. He gave a sufficient condition on the Poincaré rank p to hold his conjecture without proof, but our reduction procedure of systems in Section 3 shows that the conclusion does hold even when his sufficient condition is violated (cf. Remark in Section 3.1). This system reduction is made by *surgery operations* to get the leading term of the exponential factor of the *FFMS* (formal fundamental matrix solution). Let us explain this shortly. Let $L = (p, A(z))$ be Moser irreducible with non-zero nilpotent constant term $A_0 = A(0)$ of Jordan canonical form,

$$(1.7) \qquad A_0 = \bigoplus_{j=1}^{m_1} N_{k_j} \oplus O_{m_2}, \quad N_{k_j} \in M_{k_j}(\mathbb{C})\ (k_j \geq 2),\ O_{m_2} \in M_{m_2}(\mathbb{C}),$$

where N_{k_j} denotes the nilpotent Jordan cell of upper triangular form of rank $k_j - 1$ and O_{m_2} denotes the zero matrix. Then we define *Jordan type* $J(A_0)$ of A_0 by

$$J(A_0) := (k_1, k_2, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}. \tag{1.8}$$

The surgery operations consist of the following two type operations;

① A_0-invariant transformation by $P \in GL_N(\mathbb{C})$ (cf. Section 4.2).
② $J(A_0)$-change transformation by $P(z) \in GL_N(\mathbb{K}[z])$ (cf. Section 3.4.1).

The important fact is that under the surgery operations rankA_0 and the Moser polynomial $\mathcal{P}_A(\lambda)$ are invariant. Fundamental properties of surgery operations will be given in Section 4. As an application of A_0-invariant transformations we give a way of construction of Moser's rank reduction matrix for Moser reducible system, which is different from [Mos] and [B-P1,2].

2 Proof of Theorem 1.2

The proof will be done after several preliminary considerations. Almost statements are given without proof because of the page limitation, but they are proved by the knowledge of elementary linear algebra.

2.1 Newton polygon and Moser irreducibility

Let $A(z) \in GL_N(\mathbb{C}\{z\})$ have non-zero constant term A_0 which is not nilpotent necessarily, and $p_A(\lambda, z) = \sum_j p_j(z)\lambda^{N-j}$ be its characteristic polynomial. Then

$$p_0 \equiv 1, \quad p_j(z) = (-1)^j \times \sum\nolimits_{1 \le i_1 < \cdots < i_j \le N} \det \left(a_{i_k, i_\ell}(z)\right)_{1 \le k, \ell \le j}. \tag{2.1}$$

We define $Q(p_j) := \{(x, y) \in \mathbb{R}^2 \; ; \; x \le j, \; y \ge O(p_j)\}$. Then the Newton polygon $\mathrm{N}(A)$ of $A(z)$ is defined by

$$\mathrm{N}(A) := \mathrm{Convex-hull}\left(\bigcup\nolimits_{j=0}^{N} Q(p_j)\right). \tag{2.2}$$

Therefore, $s_0(A) := \min_{1 \le j \le N} O(p_j)/j$ in (1.6) denotes the smallest slope of sides of $\mathrm{N}(A)$ in the region $x \ge 0$. The nilpotent condition for A_0 is equivalent with that $s_0(A) > 0$.

Now, the relation between $\mathrm{N}(A)$ and Moser's (ir)reducibility is obtained by

Lemma 2.1 (relation between $\mathcal{P}_A(\lambda)$ and $\mathrm{N}(A)$) *Let* rank$A_0 = r$ (≥ 1), *where* A_0 *is not assumed to be nilpotent necessarily. Then ;*

(i) $\mathrm{N}(A)$ *lies in the region* $y \ge x - r$.

(ii) $\mathcal{P}_A(\lambda) \equiv 0$ *is equivalent that* $\mathrm{N}(A)$ *lies in the strictly upper region* $y > x - r$.

(iii) $\mathcal{P}_A(\lambda)$ *is determined by the members of monomials in* $p_A(\lambda, z)$ *on the line* $y = x - r$ *by putting* $z = 1$.

The following figure shows this relation visually.

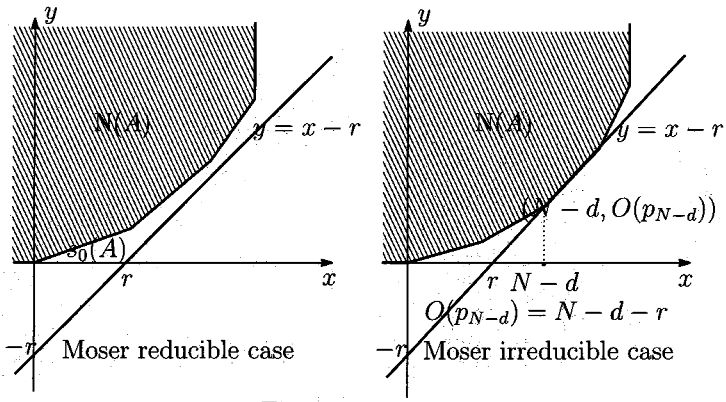

Figure 1

Lemma 2.2 (invariance of $\mathcal{P}_A(\lambda)$) *The Moser polynomial $\mathcal{P}_A(\lambda)$ is invariant under the system transformations by matrices $P(z) \in GL_N(\mathbb{C}\{z\})$. In the Moser irreducible case, let $d = \deg_\lambda \mathcal{P}_A \geq 0$. Then the vertex point $(N-d, N-d-r) \in \mathrm{N}(A)$ is the common vertex point of $\mathrm{N}(A_P)$ for all $P(z) \in GL_N(\mathbb{C}\{z\})$, and $s_0(A_P) \leq S_0(L) \leq 1$.*

2.2 Moser polynomial $\mathcal{P}_A(\lambda)$ via Moser matrix $\mathcal{A}$

We show a direct way of calculation of $\mathcal{P}_A(\lambda)$ by using a sub-matrix of A_1 in the Taylor expansion $A(z) = \sum_{n=0}^{\infty} A_n z^n$. We assume A_0 is given in nilpotent Jordan canonical form,

$$J(A_0) := (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}. \quad m_1 \geq 1. \tag{2.3}$$

The arrangement of order of $J(A_0)$ is only for the convenience. We define $\{k(j)\}_{j=0}^{m_1}$ by

$$k(0) := 0, \quad k(j) := \sum_{i=1}^{j} k_i \quad (1 \leq j \leq m_1). \tag{2.4}$$

From the coefficient matrix $A_1 = \left(a_{ij}\right)$, we choose $\{a^{[i,j]}\}_{1 \leq i,j \leq m_1+m_2}$ by

$$a^{[i,j]} := \begin{cases} a_{k(i),k(j-1)+1}, & 1 \leq i,\ j \leq m_1, \\ a_{k(i),k(m_1)+t}, & 1 \leq i \leq m_1,\ j = m_1 + t, \\ a_{k(m_1)+s,k(j-1)+1}, & i = m_1 + s,\ 1 \leq j \leq m_1, \\ a_{k(m_1)+s,k(m_1)+t}, & i = m_1 + s,\ j = m_1 + t, \end{cases} \qquad (1 \leq s, t \leq m_2). \tag{2.5}$$

Now we define *Moser's matrix* $\mathcal{A} \in M_{m_1+m_2}(\mathbb{C})$ by

$$\mathcal{A} := \left(a^{[i,j]}\right) = \begin{bmatrix} \mathcal{A}^{[1,1]} & \mathcal{A}^{[1,2]} \\ \mathcal{A}^{[2,1]} & \mathcal{A}^{[2,2]} \end{bmatrix}, \quad \mathcal{A}^{[i,j]} \in M_{m_i \times m_j}(\mathbb{C}). \tag{2.6}$$

Then the Moser polynomial $\mathcal{P}_A(\lambda)$ is calculated by

Lemma 2.3 (calculation of $\mathcal{P}_A(\lambda)$)

$$\mathcal{P}_A(\lambda) = \det\left[\{O_{m_1} \oplus \lambda I_{m_2}\} - \mathcal{A}\right] \ \left(= \sum_{j=0}^{m_2} q_j \lambda^{m_2 - j}\right). \tag{2.7}$$

This shows that $\deg_\lambda \mathcal{P}_A \le m_2$ *and typical coefficients are obtained by* $q_0 = (-1)^{m_1} \det \mathcal{A}^{[1,1]}$ *and* $q_{m_2} = (-1)^{m_1+m_2} \det \mathcal{A}$, *which may vanish.*

Remark. A similar result from different observation is seen in [B-P1, 2], where the matrix in the middle of (2.7) is called L-matrix $L(A,\lambda)$.

In actual application, it is convenient to add the Jordan type $J(A_0)$ to $\mathcal{A}$.

$$\mathcal{A} = \begin{array}{c|cccccc} & k_1 & \cdots & k_{m_1} & 1 & \cdots & 1 \\ \hline k_1 & a^{[1,1]} & \cdots & a^{[1,m_1]} & a^{[1,m_1+1]} & \cdots & a^{[1,m_1+m_2]} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ k_{m_1} & a^{[m_1,1]} & \cdots & a^{[m_1,m_1]} & a^{[m_1,m_1+1]} & \cdots & a^{[m_1,m_1+m_2]} \\ 1 & a^{[m_1+1,1]} & \cdots & a^{[m_1+1,m_1]} & a^{[m_1+1,m_1+1]} & \cdots & a^{[m_1+1,m_1+m_2]} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a^{[m_1+m_2,1]} & \cdots & a^{[m_1+m_2,m_1]} & a^{[m_1+m_2,m_1+1]} & \cdots & a^{[m_1+m_2,m_1+m_2]} \end{array}$$

It is useful to know that $\{a^{[i,j]}\}$ are the entries on the position $\boxed{*}$ in the figure below.

$A_1 =$

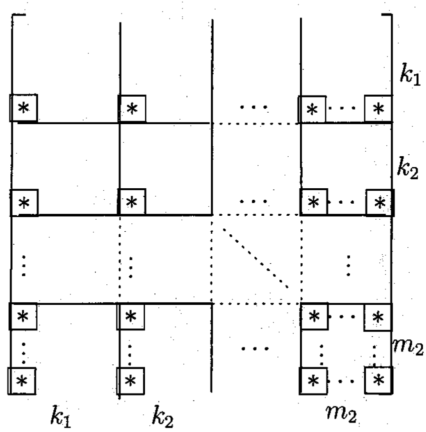

Figure 2

2.3 Summary on the irregularity $\rho(L)$ from [M-I]

For $A(z) = (a_{ij}(z)) \in M_N(\mathbb{C}\{z\})$, we put $r_{ij} = O(a_{ij}) \in \mathbb{N}_{\ge 0} \cup \{+\infty\}$, where $O(0) := +\infty$. Then *Volevič's weight* $V(A) \in \mathbb{Q}_{\ge 0} \cup \{+\infty\}$ is defined by

$$V(A) := \min_{1\le n\le N} \min_{1\le i_1<\cdots<i_n\le N} \min_{\sigma\in S_n} \frac{1}{n} \sum\nolimits_{k=1}^{n} r_{i_k, i_{\sigma(k)}} \quad (\le s_0(A)). \tag{2.8}$$

The following lemma is the most fundamental in the study of singular system $L = (p, A(z))$.

Lemma 2.4 ([Vol], [Miy1], [M-I]) *Let* $V(A) \in \mathbb{Q}_{\ge 0}$. *Then, associated with* $V(A)$, *there is a system of numbers* $T = \{t_i\}_{i=1}^N \subset \mathbb{Q}$, *which we call V-numbers, such that*

$$r_{ij} \ge t_i - t_j + V(A), \quad i,j = 1,2,\cdots,N. \tag{2.9}$$

Moreover, we can find the V-numbers $T = \{t_i\}$ satisfying the following span condition,

$$\sigma(T) := \max\{|t_i - t_j| \, ; \, i, j = 1, 2, \cdots, N\} \leq (N-1) \times V(A). \tag{2.10}$$

By this lemma, we see that our interest is in the case $V(A) < p$, since otherwise the system is regular singular at $z = 0$ (cf. [Kit], [M-I] for detail).

By Lemma 2.4, $a_{ij}(z) \in \mathbb{C}\{z\}$ are written in the form

$$a_{ij}(z) = \left\{ \mathring{a}_{ij} + o(1) \right\} z^{t_i - t_j + V(A)}, \quad \mathring{a}_{ij} = 0 \text{ if } t_i - t_j + V(A) \notin \mathbb{N}.$$

We define the principal matrix $\mathring{A}$ of $A(z)$ w.r.t. V-numbers $T = \{t_i\}$ by

$$\mathring{A} := \left(\mathring{a}_{ij} \right) \in M_N(\mathbb{C}). \tag{2.11}$$

The principal matrix $\mathring{A}$ is not determined uniquely, since T is not determined uniquely. But its eigenvalues is not. It is also important to know that by taking T which satisfy the span condition (2.10) $\mathring{A}$ is determined from the members of the Taylor coefficients $\{A_n \, ; \, 0 \leq n \leq N \times V(A)\}$. Now we give fundamental definition and results.

Definition (non-degeneracy) Let $V(A) < p$. Then we define;

(i) $L = (p, A(z))$ is called *non-degenerate in V-sense* if $V(A) = s_0(A)$ which means $\mathring{A}$ is not nilpotent.

(ii) L is called *non-degenerate of full rank* if $O(\det A(z)) = N \times V(A)$, i.e., $\det \mathring{A} \neq 0$.

Lemma 2.5 [M-I] *Let $V(A) < p$. Then we have;* (i) $\rho(L) \leq p - V(A)$.

(ii) $\rho(L) = p - V(A)$ *if and only if L is non-degenerate in V-sense.*

(iii) *If L is non-degenerate of full rank, then the leading term of the exponential factor $\Lambda(z)$ of the FFMS is obtained in the form,*

$$\operatorname{diag}\left(\cdots, \alpha_j z^{-p+V(A)}/(V(A)-p), \cdots\right), \quad \{\alpha_j\}_{j=1}^N \textit{ are the eigenvalues of } \mathring{A}. \tag{2.12}$$

Theorem 2.1 [**M-I, Th.A**] *Let $L = (p, A(z))$ be of irregular singular type. Then we can find $P(z) \in GL_N(\mathbb{C}[z])$ such that the reduced system $L_P = (p, A_P(z))$ is non-degenerate in V-sense of $V(A_P) < p$, and therefore $\rho(L) = p - V(A_P) > 0$.*

2.4 Proof of Theorem 1.2

By Theorem 2.1, we take $P(z) \in GL_N(\mathbb{C}[z])$ such that $L_P = (p, A_P(z))$ is non-degenerate in V-sense. Then by Lemma 2.2, we know that $V(A_P) = s_0(A_P) \leq S_0(L)$, which shows that $\rho(L) = \rho(L_P) = p - V(A_P) \geq p - S_0(L)$ by Lemma 2.5, (ii).

On the other hand, let $A(z)$ be the one in the theorem. For the nilpotent constant term $A(0) = A_0$ let its Jordan type be $J(A_0) = (k_1, \cdots)$, where $k_1 = min\{A_0^k = O\}$. Since the maximal size of nilpotent Jordan cells is k_1, we have $V(A) \geq 1/k_1$ by the definition (2.8) of $V(A)$. This proves that $\rho(L) \leq p - 1/k_1$ by Lemma 2.5, (i). □

3 A counterexample for Barkatou's conjecture

3.1 The case when the conjecture does hold

Before we give the counterexample for Barkatou's conjecture, we remark that for $L = (p, A(z))$ with sufficiently large p, we can conclude that $\rho(L) = p - s_0(A)$. Indeed,

Theorem 3.1. *Let $L = (p, A(z))$ be Moser irreducible with $d = \deg_\lambda \mathcal{P}_A \geq 0$, where $A_0 = A(0)$ is assumed to be a non-zero nilpotent matrix. Then it holds that*

$$(3.1) \qquad p+1 > N \times S_0(L) \quad \left(S_0(L) = \frac{N-d-r}{N-d}\right) \quad \text{implies} \quad \rho(L) = p - s_0(A).$$

Proof. If the Moser irreducible system $L = (p, A(z))$ is non-degenerate in V-sense, there is nothing to prove. So we assume that L is Moser irreducible but degenerate in V-sense. Hence, $V(A) < s_0(A) \leq S_0(L)$, where $S_0(L)$ is invariant for every reduced matrix $A_P(z)$ by $P(z) \in GL_N(\mathbb{C}[z])$ and $s_0(A_P) \leq S_0(L)$. By Theorem 2.1, we can find a matrix $P(z) \in GL_N(\mathbb{C}[z])$ such that the reduced system $L_P = (p, A_P(z))$ is non-degenerate in V-sense for which the Moser polynomial is invariant. Then we have $\rho(L) = p - s_0(A_P)$. Therefore we have to prove that $s_0(A_P) = s_0(A)$ under the assumption $p+1 > N \times S_0(L)$.

For this purpose, we recall the construction of the principal coefficient $\overset{\circ}{A}$ of $A(z)$. As mentioned before, this is determined by the members of the Taylor coefficients $\{A_n\ ;\ 0 \leq n \leq N \times V(A)\}$. The system is degenerate in V-sense if and only if $\overset{\circ}{A}$ is a nilpotent matrix, and the reduction procedure is carried out by reduction matrices in $GL_N(\mathbb{C}[z])$ obtained by using the null-vectors of the principal coefficient (cf. [M-I] for detail). This shows that, through out the reduction procedure, the operations are done by using the members of Taylor coefficients of z^n of the intermediate coefficient matrix such that

$$(3.2) \qquad 0 \leq n \leq N \times S_0(L), \quad ((\because)\ V(A) \leq s_0(A) \leq S_0(L)).$$

Let $Q(z) \in GL_N(\mathbb{C}[z])$ be an intermediate reduction matrix. Note that the reduced matrix is given by

$$A_Q(z) = Q^{-1}AQ - z^{p+1}Q^{-1}Q', \quad V(A_Q) \leq s_0(A_Q) \leq S_0(L).$$

From this expression we know that if $p+1 > N \times S_0(L)$, the term $z^{p+1}Q^{-1}Q'$ has no influence in the determination of $s_0(A_Q)(\leq S_0(L))$. This shows that $s_0(A_Q) = s_0(Q^{-1}AQ) = s_0(A)$. By continuing this procedure we finally get a reduction matrix $P(z) \in GL_N(\mathbb{C}[z])$ into a non-degenerate system in V-sense, and hence $s_0(A_P) = s_0(A)$. □

Remark. M. Barkatou gave a theorem [Bar, p.33] without proof that

(B) *If $p \geq (r+1)S_0(L)$, then it holds that $\rho(L) = p - s_0(A)$*

Our counterexamples given below do not satisfy his assumption throughout system reductions by surgery operations, but we know that the conclusion does hold after the one step reduction. Our counterexample shows that if one wants to get a condition under which we have $\rho(L) = p - s_0(A)$, we have to find a condition under which $s_0(A)$ is invariant by system reductions.

3.2 Introduction of the counterexample

First we remark that the actual calculations below were done by Mathematica 9.0.

We consider a system $L = (1, A(z))$ $(p = 1)$ of $A(z) \in M_9(\mathbb{C}[z])$ given by

$$A(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \boxed{0} & z & 0 & z & 0 & \boxed{z} & 0 & \boxed{0} & \boxed{0} \\ 0 & z^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{0} & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{z} & \boxed{0} \\ \boxed{0} & 0 & 0 & 0 & 0 & \boxed{0} & -z & \boxed{0} & \boxed{z} \\ \boxed{z} & 0 & 0 & 0 & 0 & \boxed{0} & 0 & \boxed{0} & \boxed{0} \end{pmatrix}, \qquad \begin{aligned} &V(A) = \frac{1}{4} < \frac{4}{9} = s_0(A), \\ &J(A_0) = (5, 2, 1, 1). \end{aligned}$$

In fact, the characteristic polynomial is given by

$$p_A(\lambda, z) = \lambda^9 - z\lambda^8 + (-2z + z^2)\lambda^7 - z^3\lambda^6 + (z^2 - 2z^3)\lambda^5 + z^4\lambda^3 - z^4,$$

which shows that the Newton polygon $\mathrm{N}(A)$ has only one side $\overline{(0,0),(9,4)}$ of slope $s_0(A) = 4/9$. Therefore, the system is degenerate in V-sense. Moreover, the system is Moser irreducible, since the Moser matrix and Moser polynomial are given by

$$\mathcal{A} = \begin{array}{c|cccc} & 5 & 2 & 1 & 1 \\ \hline 5 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array}, \quad \mathcal{P}_A(\lambda) = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = -1, \quad d = \deg_\lambda \mathcal{P}_A = 0.$$

Hence $S_0(L) := (N - d - r)/(N - d) = 4/9 = s_0(A) > V(A) = 1/4$, and $\rho(L) < 1 - 1/4$.

Now we shall show the following,

Statement. *The irregularity $\rho(L)$ is given by*

$$\rho(L) = 1 - \frac{2}{5} > 1 - s_0(A) = 1 - \frac{4}{9}. \tag{3.3}$$

The leading term of the exponential factor $\Lambda(z)$ of the FFMS is given by

$$\mathrm{diag}\left(\cdots, \frac{\alpha_j}{-3/5} z^{-3/5}, \cdots, \frac{\beta_k}{-1/2} z^{-1/2}, \cdots\right), \quad \alpha_j^5 + 1 = 0, \quad \beta_k^4 - 1 = 0. \tag{3.4}$$

3.3 Reduction into a non-degenerate system in V-sense

For the proof of (3.3) we reduce the system into a non-degenerate one in V-sense. It is done by the following matrix of $J(A_0)$-change transformation,

$$P_1(z) = \mathrm{diag}(1, 1, 1, 1, z, 1, 1, 1, 1) \circ E(5, 1; 1) \circ E(5, 3; 1),$$

where $E(i,j;c) \in M_9(\mathbb{C})$ differs from the identity matrix I_9 by c on the (i,j)-position. The meaning of this transformation matrix will be learned in Section 3.4.1 below.

For the reduced system $L_1 = (1, A_1(z))$, $A_1(z)$ and its Moser matrix $\mathcal{A}_1$ become

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & 0 & z & 0 & \boxed{z} & 0 & 0 & \boxed{0} & \boxed{0} \\ -z-z^2 & -z & -z & 0 & -z & 1 & 0 & 0 & 0 \\ 0 & z^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \boxed{0} & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{z} & \boxed{0} \\ \boxed{0} & 0 & 0 & 0 & \boxed{0} & 0 & -z & \boxed{0} & \boxed{z} \\ \boxed{z} & 0 & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{0} & \boxed{0} \end{pmatrix}, \quad \mathcal{A}_1 = \begin{array}{c|cccc} & 4 & 3 & 1 & 1 \\ \hline 4 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{array},$$

for which we have $V(A_1) = 2/5 = s_0(A_1)$ and $\mathcal{P}_{A_1}(\lambda) = -1$. This means that the reduced system L_1 is Moser irreducible and non-degenerate in V-sense. Indeed, the characteristic polynomial $p_{A_1}(\lambda, z) = \sum_{j=0}^{9} p_j(z)\lambda^{9-j}$ is obtained by

$$\{p_j(z)\}_{j=0}^{9} = \{\underline{1}, 0, -2z, -z^2, -z^4 - 3z^3 + z^2, \underline{z^2} - z^4, z^4 - z^5, z^4, 0, \underline{-z^4}\}.$$

This shows that the Newton polygon $\mathrm{N}(A_1)$ has two sides $\overline{(0,0),(5,2)}$ of slope $2/5 = s_0(A_1) < s_0(A)$ and $\overline{(5,2),(9,4)}$ of slope $1/2$.

This proves (3.3), since L_1 is non-degenerate in V-sense of $V(A_1) = 2/5$.

In order to prove (3.4), it is convenient to explain the Puiseux expansion of the eigenvalues of $A_1(z)$, i.e., the roots of $p_{A_1}(\lambda, z) = 0$. By the property of $\mathrm{N}(A_1)$, the leading terms of the Puiseux expansions are obtained from $\lambda^9 + z^2\lambda^4 - z^4 = 0$. Or, equivalently, they are obtained from $\lambda^5 + z^2 = 0$ and $\lambda^4 - z^2 = 0$.

This is the reason why the form in (3.4) appears, but we need a concrete proof by reducing $L_1 = (1, A_1(z))$ into a decomposable form by non-degenerate subsystems of full rank, which is done by the surgery operations which is explained in the following subsection.

3.4 Reduction of L_1 by surgery operations

3.4.1 What is $J(A_0)$-change transformation ?

In our system reduction developed below, we adopt surgery operations which consist of

1. A_0-invariant transformation by $P \in GL_N(\mathbb{C})$ (cf. Section 4.2 for detail).

2. $J(A_0)$-change transformation by $P(z) \in GL_N(\mathbb{K}[z])$ defined below which preserves the Moser polynomial $\mathcal{P}_A(\lambda)$ (see Example below and Theorem 4.1).

Let us explain $J(A_0)$-change transformation which is employed in this paper.

Let $L = (p, A(z))$ be a Moser irreducible system with $J(A_0) = (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}$. The $J(A_0)$-change transformation is taken only when the Moser matrix $\mathcal{A} = \left(a^{[i,j]}\right)$ has a row vector, say the i_0-row vector of $1 \le i_0 \le m_1$, such that on the row there is only one non-zero element, say $a^{[i_0,j_0]} \neq 0$. Then we can find a reduced system such that the constant term of the reduced matrix has the Jordan type

(3.5) $\quad (\cdots, k_{j_0} + 1, \cdots, k_{i_0} - 1, \cdots)$, *the others are same with those in* $J(A_0)$.

In fact, we first make a transformation by

$$Q(z) = D_N(k(i_0); a^{[i_0,j_0]}z) := \mathrm{diag}\left(1, \cdots, 1, a^{[i_0,j_0]}\, z, 1, \cdots, 1\right) \in GL_N(\mathbb{K}[z]),$$

where $a^{[i_0,j_0]}\, z$ is located on the diagonal position $k(i_0)$. Let $A_Q(0)$ be the constant term of the reduced matrix of the reduced system $L_P = (p, A_Q(z))$. Then $A_Q(0)$ differs from A_0 by that $(k(i_0)-1, k(i_0))$ entry vanishes, $(k(i_0), k(j_0-1)+1)$ entry is 1, and other non-zero elements may appear on the positions $(k(i_0), j)$ such that $k(i-1)+1 < j \le k(i)$ with $1 \le i \le m_1$. This shows that $\mathrm{rank} A_Q(0) = \mathrm{rank} A_0$. The last non-zero elements are killed by using the 1's on the off-diagonal positions of the Jordan form A_0. Then, by making an arrangement of order if necessary, we finally obtain a reduced system of the desired form (3.5) by a matrix $P(z) = Q(z)R$ of $R \in GL_N(\mathbb{C})$ (cf. $P_1(z)$ in Section 3.3).

Example. The assumption that $1 \le i_0 \le m_1$ is posed for preserving the rank of the constant term of the reduced matrix. For example, let us consider a Moser irreducible system $L = (p, A(z))$ of $A(z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & z \\ z & 0 & 0 \end{pmatrix}$ for which $J(A_0) = (2,1)$, $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{P}_{\mathcal{A}}(\lambda) = -1$. If we make a system transformation by $P(z) = \mathrm{diag}(1, z, 1)$, we get $A_P(z) = \begin{pmatrix} 0 & z & 0 \\ 0 & -z^p & 1 \\ z & 0 & 0 \end{pmatrix}$ for which $J(A_P(0)) = (1,2)$ and $\mathcal{A}_P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. But if we make a system transformation by $Q(z) = \mathrm{diag}(1,1,z)$, we get a Moser reducible system $L_Q = (p, A_Q(z))$ of $A_Q(z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & z^2 \\ 1 & 0 & -z^p \end{pmatrix}$, which is reduced by an arrangement of the order into $A_R(z) = \begin{pmatrix} -z^p & 1 & 0 \\ 0 & 0 & 1 \\ z^2 & 0 & 0 \end{pmatrix}$ for which $J(A_R(0)) = (3) (= (3,0))$ and $\mathcal{A}_R = (0)$.

3.4.2 First reduction by $J(A_0)$-change transformation

We make a $J(A_0)$-change transformation for $L_1 = (1, A_1(z))$ by

$$P_2(z) = \mathrm{diag}(1,1,1,1,1,\boxed{z},\boxed{z},1,1) = D_9(6;z) \circ D_9(7;z).$$

By this matrix, $J(A_0)$-change transformation is continued twice (cf. Section 3.4.1).

Then for the reduced system $L_2 = (1, A_2(z))$, $A_2(z)$, its Moser matrix $\mathcal{A}_2$ and its Moser polynomial become

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\boxed{0} & 0 & z & 0 & \boxed{z} & \boxed{0} & 0 & 0 & \boxed{0} \\
\boxed{-z-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z} & 0 & 0 & \boxed{0} \\
0 & z & 0 & 0 & 0 & -z & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -z & 1 & 0 \\
\boxed{0} & 0 & 0 & 0 & \boxed{0} & \boxed{0} & -z^2 & 0 & \boxed{z} \\
\boxed{z} & 0 & 0 & 0 & \boxed{0} & \boxed{0} & 0 & 0 & \boxed{0}
\end{pmatrix}, \quad
\begin{array}{c|cccc}
 & 4 & 1 & 3 & 1 \\ \hline
4 & 0 & 1 & 0 & 0 \\
1 & \underline{-1} & -1 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 \\
1 & \boxed{1} & 0 & 0 & 0
\end{array},$$

$$\mathcal{P}_{A_2}(\lambda) = -1.$$

3.4.3 Second reduction by A_0-invariant transformation

We kill 1 on $(4,1)$-position of $\mathcal{A}_2$ by the second row by the following A_0-invariant matrix,

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{-1} & 0 & 0 & 0 & 1 \end{pmatrix} \underset{put}{=} \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & I_{4\times4} & 0 & 0 & 0 \\ 1 & 0 & I_{1\times1} & 0 & 0 \\ 3 & 0 & 0 & I_{3\times3} & 0 \\ 1 & 0 & -I_{1\times1} & 0 & I_{1\times1} \end{array},$$

where $I_{m\times n}$ denotes a rectangular matrix of type $m \times n$ defined by $I_{m\times n} := (\delta_{ij})$ if $m \geq n$ and $I_{m\times n} := (\delta_{m-i,n-j})$ if $m \leq n$, by Kronecker's delta δ_{ij}.

Then for the reduced system $L_3 = (1, A_3(z))$, $A_3(z)$ and its Moser matrix $\mathcal{A}_3$ become

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{0} & 0 & z & 0 & \boxed{z} & \boxed{0} & 0 & 0 & \boxed{0} \\ \boxed{-z-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z} & 0 & 0 & \boxed{0} \\ 0 & z & 0 & 0 & 0 & -z & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1 & 0 \\ \boxed{0} & 0 & 0 & 0 & \boxed{-z} & \boxed{0} & -z^2 & 0 & \boxed{z} \\ \boxed{-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z} & 0 & 0 & \boxed{0} \end{pmatrix}, \quad \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & 0 & 1 & 0 & 0 \\ 1 & \underline{-1} & -1 & \boxed{1} & 0 \\ 3 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{array}.$$

See Lemmas 4.1 and 4.2 for the relation between the A_0-invariant transformation and the transformation of Moser matrix.

3.4.4 Third reduction by A_0-invariant transformation

We kill 1 on the (2,3)-position of $\mathcal{A}_3$ by the first column by

$$P_4 \underset{put}{=} \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & 1_{4\times4} & 0 & I_{4\times3} & 0 \\ 1 & 0 & I_{1\times1} & 0 & 0 \\ 3 & 0 & 0 & I_{3\times3} & 0 \\ 1 & 0 & 0 & 0 & I_{1\times1} \end{array}.$$

Then for the reduced system $L_4 = (1, A_4(z))$, $A_4(z)$ and its Moser matrix $\mathcal{A}_4$ become

$$\begin{pmatrix} 0 & 1-z & 0 & 0 & 0 & z & -z & 0 & 0 \\ 0 & z & 1 & 0 & 0 & 0 & 2z & 0 & 0 \\ z^2 & z & 0 & 1 & z & z^2 & z+z^2 & 0 & -z \\ \boxed{0} & 0 & z & 0 & \boxed{z} & \boxed{0} & 0 & z & \boxed{0} \\ \boxed{-z-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{-z^2} & -z & -z & \boxed{0} \\ 0 & z & 0 & 0 & 0 & -z & 1+z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z & 1 & 0 \\ \boxed{0} & 0 & 0 & 0 & \boxed{-z} & \boxed{0} & -z^2 & 0 & \boxed{z} \\ \boxed{-z^2} & -z & -z & 0 & \boxed{-z} & \boxed{z-z^2} & -z & -z & \boxed{0} \end{pmatrix}, \quad \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & 0 & \underline{1} & 0 & 0 \\ 1 & -1 & \boxed{-1} & 0 & 0 \\ 3 & 0 & \boxed{-1} & 0 & 1 \\ 1 & 0 & \boxed{-1} & 1 & 0 \end{array}.$$

3.4.5 Fourth reduction by A_0-invariant transformation, the final reduction

We kill -1's on the second column of $\mathcal{A}_4$ by the first row by

$$P_5 \underset{put}{=\!=} \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & I_{4\times 4} & 0 & 0 & 0 \\ 1 & -I_{1\times 1} & I_{1\times 1} & 0 & 0 \\ 3 & -I_{3\times 4} & 0 & I_{3\times 3} & 0 \\ 1 & -I_{1\times 4} & 0 & 0 & I_{1\times 1} \end{array}.$$

Then the reduced system $L_5 = (1, A_5(z))$ becomes

$$A_5(z) = \begin{pmatrix} 0 & 1-2z & z & 0 & 0 & z & -z & 0 & 0 \\ 0 & z & 1-2z & 0 & 0 & 0 & 2z & 0 & 0 \\ z^2 & z-z^2 & -z-z^2 & 1 & z & z^2 & z+z^2 & 0 & -z \\ \boxed{0} & 0 & z & -2z & \boxed{z} & \boxed{0} & 0 & z & \boxed{0} \\ \boxed{-z-z^2} & z^2-z & z & 0 & \boxed{0} & \boxed{-z^2} & -z & 0 & \boxed{0} \\ 0 & 3z & -3z & 0 & 0 & -z & 1+3z & 0 & 0 \\ z^2 & z-z^2 & -z^2 & 0 & z & z^2 & z^2 & 1 & -z \\ \boxed{0} & 0 & z+z^2 & -2z & \boxed{0} & \boxed{0} & -z^2 & z & \boxed{z} \\ \boxed{-z^2} & z^2-2z & z & 0 & \boxed{0} & \boxed{z-z^2} & -z & 0 & \boxed{0} \end{pmatrix},$$

and its Moser matrix has the following decomposition by cyclic matrices.

$$\mathcal{A}_5 = \begin{array}{c|cccc} & 4 & 1 & 3 & 1 \\ \hline 4 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array} = \begin{array}{c|cc} & 4 & 1 \\ \hline 4 & 0 & 1 \\ 1 & -1 & 0 \end{array} \oplus \begin{array}{c|cc} & 3 & 1 \\ \hline 3 & 0 & 1 \\ 1 & 1 & 0 \end{array}.$$

We make a blocked decomposition of $A_5(z)$ following the decomposition $9 = 5 + 4$, and the decomposition is denoted by $A_5(z) = \left(A_{i,j}(z)\right)_{i,j=1,2}$. We take the subsystems of L_5 taken from the diagonal blocks $L_{5,i} = (1, A_{i,i}(z))$ $(i = 1, 2)$,

$$A_{1,1}(z) = \begin{pmatrix} 0 & 1-2z & z & 0 & 0 \\ 0 & z & 1-2z & 0 & 0 \\ z^2 & z-z^2 & -z-z^2 & 1 & z \\ \boxed{0} & 0 & z & -2z & \boxed{z} \\ \boxed{-z-z^2} & z^2-z & z & 0 & \boxed{0} \end{pmatrix}, \quad \begin{array}{l} V(A_{1,1}) = s_0(A_{1,1}) = \dfrac{2}{5}, \\ O(\det(A_{1,1}(z)) = 2, \end{array}$$

$$A_{2,2}(z) = \begin{pmatrix} -z & 1+3z & 0 & 0 \\ z^2 & z^2 & 1 & -z \\ \boxed{0} & -z^2 & z & \boxed{z} \\ \boxed{z-z^2} & -z & 0 & \boxed{0} \end{pmatrix}, \quad \begin{array}{l} V(A_{2,2}) = s_0(A_{2,2}) = \dfrac{1}{2}, \\ O(\det A_{2,2}(z)) = 2. \end{array}$$

These show that the systems $L_{5,i}$ are non-degenerate of full rank. By Lemma 2.5, (iii) the leading term of the exponential factor of the FFMS for each $L_{5,i}$ is obtained by those in (3.4).

Remark. By taking V-numbers for $A_{i,i}(z)$, we make a system reduction by $P_6(z) = \operatorname{diag}(1, z^{2/5}, z^{4/5}, z^{6/5}, z^{3/5}, 1, z^{1/2}, z, z^{1/2})$. The reduced system becomes $L_6 = (1, A_6(z))$ of $A_6(z) = (\overset{\circ}{A}_1 z^{2/5} \oplus \overset{\circ}{A}_2 z^{1/2})$ + higher order term, with $\overset{\circ}{A}_i$ of the principal matrices of $A_{i,i}(z)$, respectively. This reduced system shows the expression (3.4) (see also [Miy2, Sec.4]).

4 Properties of surgery operations

4.1 Invariance of Moser polynomial under surgery operations

Theorem 4.1. *The Moser polynomial $\mathcal{P}_A(\lambda)$ is invariant under the surgery operations defined in Section 3.4.1 for the Moser irreducible system $L = (p, A(z))$.*

Proof. The conclusion is obvious for A_0-invariant transformation, since it is done by matrices in $GL_N(\mathbb{C})$. Then we prove the invariance under the $J(A_0)$-change transformation in Section 3.4.1. Recall that it is done firstly by a diagonal matrix $Q(z) = \operatorname{diag}(1, \cdots, 1, cz, 1, \cdots, 1)$ ($c \neq 0$, $1 \leq i_0 \leq m_1$), where cz is located on the position $k(i_0)$, where $J(A_0) = (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}$. Then our $J(A_0)$-change transformation is completed by a matrix of the form $P(z) = Q(z)R$ with an invertible constant matrix R (cf. Section 3.4.1 for detail). Therefore, $A_P(z) = R^{-1}A_Q(z)R$ shows that $\mathcal{P}_{A_P}(\lambda) = \mathcal{P}_{A_Q}(\lambda)$, and hence it is enough to prove $\mathcal{P}_{A_Q}(\lambda) = \mathcal{P}_A(\lambda)$.

Before we start the proof, recall that by the above $J(A_0)$-change transformation we have $J(A_P(0)) = (\cdots, k_{j_0}+1, \cdots, k_{i_0}-1, \cdots)$ and $\operatorname{rank} A_0$ is preserved. Therefore, from the form of $Q(z)$, we know that $\operatorname{rank} A_Q(0) = \operatorname{rank} A_0$ (cf. Section 3.4.1). The reduced matrix $A_Q(z)$ is expressed in the following form,

$$\begin{aligned} A_Q(z) &= Q(z)^{-1}A(z)Q(z) - z^{p+1}Q^{-1}(z)Q'(z) \\ &= Q^{-1}(z)\left\{A(z) - \operatorname{diag}(0, \cdots, 0, z^p, 0, \cdots, 0)\right\} Q(z) \underset{put}{=} Q^{-1}(z)\,\tilde{A}_Q(z)\,Q(z). \end{aligned}$$

This shows that $p_{A_Q}(\lambda, z) = p_{\tilde{A}_Q}(\lambda, z)$ for the characteristic polynomials. Furthermore, the expression shows that $\operatorname{rank}\tilde{A}_Q(0) = \operatorname{rank} A_0$, and hence we have $\operatorname{rank}\tilde{A}_Q(0) = \operatorname{rank} A_Q(0)$. Then by Lemma 2.1, (iii) we have $\mathcal{P}_{A_Q}(\lambda) = \mathcal{P}_{\tilde{A}_Q}(\lambda)$. On the other hand, the above expression shows $\tilde{A}_Q = \mathcal{A}$, since $J(\tilde{A}_Q(0)) = J(A_0)$. Then by Lemma 2.3 we conclude that $\mathcal{P}_{\tilde{A}_Q}(\lambda) = \mathcal{P}_A(\lambda)$. This shows that $\mathcal{P}_{A_Q}(\lambda) = \mathcal{P}_A(\lambda)$ as desired. □

4.2 A_0-invariant transformation and Moser's rank reduction

We study the relation between the transformation of Moser's matrix and the associated A_0-invariant transformation of $A(z)$ for $L = (p, A(z))$ with $J(A_0) = (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}$.

An elementary matrix $\mathcal{E}_{m_1+m_2}(i, j; c) \in M_{m_1+m_2}(\mathbb{C})$ $(i \neq j)$ is defined by

(*) $\mathcal{E}_{m_1+m_2}(i, j; c)$ differs from $I_{m_1+m_2}$ (identity matrix) with c on the (i, j)-position.

We omit to write $m_1 + m_2$ when it is obviously known. Let $I_{k_i \times k_j}$ be a rectangular matrix of type $k_i \times k_j$ $(k_i := 1$ if $i > m_1)$ defined by

$$I_{k_i \times k_j} = \left(\delta_{\ell,m}\right)_{\ell,m} \quad (k_i \geq k_j), \qquad I_{k_i \times k_j} = \left(\delta_{k_i-l,k_j-m}\right)_{\ell,m} \quad (k_i \leq k_j), \tag{4.1}$$

where $\delta_{\ell,m}$ denotes Kronecker's delta. Then the associated A_0-invariant matrix $E(i, j; c)$ with $\mathcal{E}(i, j; c)$, which is blocked decomposed form with that of $J(A_0)$, is defined by

(*) $E(i, j; c)$ differs from I_N $(N = \sum_{j=1}^{m_1} k_j + m_2)$ with $cI_{k_i \times k_j}$ on the (i, j) block.

Then we can prove the following

Lemma 4.1. *Let $L_E := (p, A_E(z))$ be the reduced system of A_0-invariant transformation by $E = E(i, j; c)$. Then the Moser matrix $\mathcal{A}_E$ of $A_E(z)$ is obtained by*

(i) *If $k_i > k_j$, then $\mathcal{A}_E = \mathcal{E}^{-1}(i, j; c)\, \mathcal{A}$, where $\mathcal{E}^{-1}(i, j; c) = \mathcal{E}(i, j; -c)$.*

(ii) *If $k_i = k_j$, then $\mathcal{A}_E = \mathcal{E}^{-1}(i, j; c)\, \mathcal{A}\, \mathcal{E}(i, j; c)$.*

(iii) *If $k_i < k_j$, then $\mathcal{A}_E = \mathcal{A}\, \mathcal{E}(i, j; c)$.*

The properties (i) and (iii) imply that

Lemma 4.2. (a) *Let $a^{[i_0,j_0]} \neq 0$ in $\mathcal{A}$, and suppose $k_{i_0} > k_i$. Then by $\mathcal{E}(i, i_0; c)$ with $c = a^{[i,j_0]}/a^{[i_0,j_0]}$, we can kill (i, j_0) entry in $\mathcal{A}_E = \mathcal{E}^{-1}(i, i_0; c)\, \mathcal{A}$.*

(c) *Let $a^{[i_0,j_0]} \neq 0$ in $\mathcal{A}$, and suppose $k_{j_0} > k_j$. Then by $\mathcal{E}(j_0, j; c)$ with $c = -a^{[i_0,j]}/a^{[i_0,j_0]}$, we can kill (i_0, j) entry in $\mathcal{A}_E = \mathcal{A}\, \mathcal{E}(j_0, j; c)$.*

By applying this lemma we prove a characterization of Moser reducible system by

Theorem 4.2. *A system $L = (p, A(z))$ with $J(A_0) = (k_1, \cdots, k_{m_1}, 1, \cdots, 1) \in \mathbb{N}^{m_1+m_2}$ is Moser reducible if and only if by an A_0-invariant transformation the Moser matrix is reduced into a form, which is written again by $\mathcal{A} = \left(a^{[i,j]}\right)$, so that there are two numbers (s_0, t_0) of $s_0 \leq m_1 \leq t_0 \leq m_1 + m_2$ such that the row vectors $\{\vec{a}_i\}$ of $\mathcal{A}$ satisfy;*

- *$\vec{a}_{s_0} = (0, \cdots, 0, a^{[s_0,t_0+1]}, \cdots, a^{[s_0,m_1+m_2]})$,*
- *For $t_0 + 1 \leq i \leq m_1 + m_2$, $\vec{a}_i = (0, \cdots, 0, a^{[i,i+1]}, \cdots, a^{[i,m_1+m_2]})$.*

To this reduced system, the Moser's rank reduction is done by the matrix

$$\mathrm{diag}(1, \cdots, 1, z, 1, \cdots, 1, z, \cdots, z) \tag{4.2}$$

where z's are located on the position i of $i = s_0$ and of $t_0 + 1 \leq i \leq m_1 + m_2$.

Proof. We only give a proof of the necessity, since the converse is obvious from the determination of Moser polynomial by Lemma 2.3. Let $L = (p, A(z))$ be Moser reducible with non-zero nilpotent $A_0 = A(0)$. For the Jordan type $J(A_0) \in \mathbb{N}^{m_1+m_2}$ we assume that $k_1 \geq k_2 \geq \cdots \geq k_{m_1}$ without loss of generality.

The Moser reducibility condition $\mathcal{P}_A(\lambda) \equiv 0$ implies that $\det \mathcal{A} = 0$. We take a left null vector $\vec{\ell} = (\ell_1, \cdots, \ell_{i_0-1}, 1, 0, \cdots, 0)$ of $\mathcal{A}$, i.e., $\vec{\ell}\mathcal{A} = \vec{0}$. We define

$$\mathcal{E}(\vec{\ell}) := \prod\nolimits_{i=1}^{i_0-1} \mathcal{E}(i_0, i; -\ell_i), \quad E(\vec{\ell}) := \prod\nolimits_{i=1}^{i_0-1} E(i_0, i; -\ell_i).$$

Then the reduced Moser matrix $\mathcal{A}_{E(\vec{\ell})}$, the i_0-th row vector vanishes, is obtained by

$$\mathcal{A}_{E(\vec{\ell})} = \mathcal{E}(\vec{\ell})^{-1} \mathcal{A} \prod\nolimits_{i=i_0} \mathcal{E}(i_0, i; -\ell_i).$$

We write it again by $\mathcal{A}$ for the simplicity of the description below.

If $1 \leq i_0 \leq m_1$ we stop the reduction procedure here. If $i_0 > m_1$, we may assume $i_0 = m_1 + m_2$ by changing the arrangement of order. We define the matrix $\mathcal{A}_1$ of size $m_1 + m_2 - 1$ obtained from $\mathcal{A}$ by removing the last row and column. By the Moser reducibility condition $\mathcal{P}_A(\lambda) \equiv 0$ we easily see that $\det \mathcal{A}_1 = 0$ (cf. Lemma 2.3). Then by applying the above operation for $\mathcal{A}_1$, we can conclude that there is an i_1 ($\leq m_1 + m_2 - 1$) row vector of $\mathcal{A}_1$ which vanishes. Then according to the position of i_1 as above, we stop or continue the similar procedure. The Moser reducibility assumption $\mathcal{P}_A(\lambda) \equiv 0$ allows us to continue the procedures until we get the desired form.

The matrix form (4.2) for Moser's rank reduction is easily seen. $\square$

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